

STABILITY OF PERIODIC TRAVELING WAVES FOR NONLINEAR DISPERSIVE EQUATIONS

VERA MIKYOUNG HUR AND MATHEW A. JOHNSON

ABSTRACT. We consider the stability and instability of periodic traveling waves for Korteweg-de Vries type equations with fractional dispersion and other nonlinear dispersive equations. We establish that a constrained minimizer for the related variational problem is nonlinearly stable to period preserving perturbations. We also discuss when the associated linearized equation admits exponentially growing solutions. The proof utilizes the variational structure of the equation.

1. INTRODUCTION

We study the stability and instability of periodic traveling waves for a class of nonlinear dispersive equations in one spatial dimension, in particular, equations of Korteweg-de Vries (KdV) type

$$(1.1) \quad u_t - \mathcal{M}u_x + f(u)_x = 0$$

in the theory of wave motion. Here $t \in \mathbb{R}$ is typically proportional to elapsed time and $x \in \mathbb{R}$ is usually related to the spatial variable in the primary direction of wave propagation; $u = u(x, t)$ is real valued, frequently representing the wave profile or a velocity. Throughout we express partial differentiation either by a subscript or using the symbol ∂ . Moreover \mathcal{M} is a Fourier multiplier defined as $\widehat{\mathcal{M}u}(\xi) = m(\xi)\hat{u}(\xi)$, characterizing dispersion in the linear motion, while f determines nonlinearity. In many examples of interest f obeys a power law.

Perhaps the best known among equations of the form (1.1) is the KdV equation

$$u_t + u_{xxx} + (u^2)_x = 0$$

itself, which was first put forward in [Bou77] and [KdV95] to model the unidirectional propagation of surface water waves of small amplitudes and long wavelengths in a channel; it has since found relevances in other physical contexts such as Fermi-Pasta-Ulam lattices. Observe however that (1.1) defines a *nonlocal* equation unless the dispersion symbol m is a polynomial of $i\xi$. Examples with nonlocal dispersion include the Benjamin-Ono (BO) equation (see [Ben67] and [Ono75]) and the intermediate long wave equation (see [Jos77]), for which $m(\xi) = |\xi|$ and $\xi \coth \xi - 1$, respectively, while $f(u) = u^2$. Another interesting example corresponds to $m(\xi) = \sqrt{(\tanh \xi)/\xi}$ and $f(u) = u^2$, which Whitham in [Whi74] proposed to describe “breaking” of surface water waves.

Date: March 21, 2013.

2010 Mathematics Subject Classification. 35B35, 35Q53, 35B10.

Key words and phrases. stability; periodic waves; nonlinear dispersive equation; nonlocal.

A traveling wave solution of (1.1) takes the form $u(x, t) = u(x - ct)$, where $c \in \mathbb{R}$ and u satisfies by quadrature that

$$(1.2) \quad \mathcal{M}u - f(u) + cu + a = 0$$

for some $a \in \mathbb{R}$. In other words, it steadily translates at a constant speed without changing the configuration. The KdV equation itself is well known to admit periodic traveling waves expressible in closed form, called cnoidal waves (see [KdV95], for example). Moreover Benjamin in [Ben67] explicitly calculated periodic traveling waves of the BO equation. For a broad class of dispersion operators and nonlinearities, a plethora of periodic traveling waves of (1.1) may be attained via variational arguments. To illustrate this, we will discuss in Section 2 a minimization problem for a family of KdV type equations with fractional dispersion.

Benjamin in his seminal work [Ben72] (see also [Bon75]) demonstrated that solitary waves of the KdV equation are nonlinearly stable. By a solitary wave, incidentally, we mean a traveling wave solution which vanishes asymptotically. Benjamin's proof hinges upon the observation that the so-called KdV soliton arises as a constrained minimizer for a suitable variational problem, namely a ground state, and utilizes spectral properties of the associated linearized operator. The proof was subsequently developed into a powerful stability theory in [GSS87] for abstract Hamiltonian systems and led to numerous applications. In the case of $m(\xi) = |\xi|^\alpha$, $\alpha \geq 1$, and $f(u) = u^{p+1}$, $p \geq 1$, in (1.1), in particular, solitary waves were shown in [BSS87] (see also [SS90, Wei87]) to arise as energy minimizers subject to constant momentum and to be nonlinearly stable if $p < 2\alpha$, whereas they are constrained energy saddles and nonlinearly unstable if $p > 2\alpha$.

We will take matters further in Section 4 and establish that a periodic traveling wave of a KdV equation with fractional dispersion is nonlinearly stable to period preserving perturbations, provided that it arises as an energy minimizer constrained to conservations of the momentum and the mass. For (local) KdV equations with general nonlinearities, henceforth called generalized KdV equations, the nonlinear stability of a periodic traveling wave to the same period perturbations was determined in [Joh09] through spectral conditions, which were effectively expressed in terms of eigenvalues of the associated monodromy map (the periodic Evans function): see also [AP07, APBS06, BJK11, DK10, DN10]. Confronted with nonlocal operators, however, spectral problems may be out of reach by Evans function techniques. We will instead make an effort to replace ODE based approaches by functional analysis ones. The program was recently set out in [BH12]; see also [Lin08, Joh12].

As a key intermediate step we will demonstrate in Section 3 that the linearized operator associated with (1.2) is *nondegenerate* at a periodic constrained minimizer for a KdV equation with fractional dispersion. That is to say, its kernel is merely generated by spatial translations. The nondegeneracy of the linearization proves a spectral condition, which plays a central role in the stability of traveling waves (see [Wei87, Lin08] among others) and the blowup analysis (see [KMR12], for example) for the related time evolution equation, and therefore it is of independent interest. For (local) generalized KdV equations, the nondegeneracy at a periodic traveling wave was identified in [Joh09] with the requirement that the wave amplitude not be a critical point of the period, and the property was verified in [Kwo89], for example, at ground states in all dimensions. These proofs utilize shooting arguments and the Sturm-Liouville theory for ODEs, and unfortunately they may not be applicable to

nonlocal operators. Nevertheless, Frank and Lenzmann in their recent contribution [FL12] established the nondegeneracy at ground states for a family of nonlinear nonlocal equations, which we will follow. The idea is to find a suitable substitute for the Sturm-Liouville theory to estimate the number of sign changes in eigenfunctions for linear operators involving fractional derivatives.

The present development may readily be adapted to other nonlinear dispersive equations. We will illustrate this in Section 5 by discussing equations of regularized long wave type. Finally, we will remark in Section 6 about Lin's recent approach in [Lin08] to linear instability.

2. EXISTENCE OF CONSTRAINED ENERGY MINIMIZERS

We shall address the stability and instability mainly for the Korteweg-de Vries (KdV) equation with fractional dispersion

$$(2.1) \quad u_t - \Lambda^\alpha u_x + (u^2)_x = 0,$$

where $0 < \alpha \leq 2$ and $\Lambda = \sqrt{-\partial_x^2}$ is defined via the Fourier transform as $\widehat{\Lambda u}(\xi) = |\xi|\widehat{u}(\xi)$, describing fractional derivatives. For $0 < \alpha < 1$, alternatively,

$$\Lambda^\alpha u(x) = C(\alpha) PV \int_{-\infty}^{\infty} \frac{u(x) - u(y)}{|x - y|^{1+\alpha}} dy,$$

where PV stands for the Cauchy principal value and $C(\alpha)$ is an appropriate normalization constant.

In the case of $\alpha = 2$, notably, (2.1) recovers the KdV equation while in the case of $\alpha = 1$ it corresponds to the BO equation. In the case* of $\alpha = -1/2$, moreover, (2.1) was shown in [Hur12] to approximate up to quadratic order the surface water wave problem in two spatial dimensions in the infinite depth case. As indicated in the introduction, (2.1) is nonlocal for $0 < \alpha < 2$. Incidentally, fractional powers of the Laplacian occur in many physical systems including quantum mechanics [LY87] and wave turbulence [MMT97], as well as in finances [CSS08].

The present treatment may be adapted mutatis mutandis to general power-law nonlinearities (see Remark 2.3, for example). We focus on the quadratic nonlinearity, however, to simplify the exposition. Furthermore the quadratic nonlinearity is characteristic of many wave phenomena; we refer the reader to [Whi74].

Throughout we will work in the periodic, L^2 -based Sobolev space setting over the interval $[0, T]$, where $T > 0$ is fixed; but at times it is treated as a free parameter. We define a periodic Sobolev space of fractional order as a subspace of $L_{\text{per}}^2([0, T])$ equipped with the norm

$$\|u\|_{H_{\text{per}}^{\alpha/2}([0, T])}^2 = \int_0^T (u^2 + u\Lambda^\alpha u) dx,$$

where $0 < \alpha < 2$. Throughout, we employ the standard notation $\langle \cdot, \cdot \rangle$ for the inner product on $L_{\text{per}}^2([0, T])$.

Notice that (2.1) can be written as a Hamiltonian system

$$u_t = J\delta H(u),$$

* Note that $\Lambda^\alpha \partial_x$ is non singular for $\alpha \geq -1$.

where $J = \partial_x$ is the (singular) symplectic form,

$$(2.2) \quad H(u) = \int_0^T \left(\frac{1}{2} u \Lambda^\alpha u - \frac{1}{3} u^3 \right) dx$$

is the Hamiltonian, interpreted as the energy, and δ denotes variational differentiation. Notice that (2.1) possesses in addition to H two conserved quantities

$$(2.3) \quad P(u) = \int_0^T \frac{1}{2} u^2 dx$$

and

$$(2.4) \quad M(u) = \int_0^T u dx,$$

which we refer to as the momentum and the mass, respectively. The conservation of P bears out that (2.1) is invariant under spatial translations thanks to the Noether theorem, while M is a Casimir invariant of the flow induced by (2.1) and is associated with the fact that the kernel of the symplectic form is spanned by a constant. It is readily verified that

$$(2.5) \quad \delta P(u) = u \quad \text{and} \quad \delta M(u) = 1.$$

Moreover (2.1) enjoys the scaling symmetry under

$$(2.6) \quad u(x, t) \mapsto \lambda^\alpha u(\lambda x, \lambda^{\alpha+1} t)$$

for any $\lambda > 0$.

Remark 2.1 (Well-posedness). In the range $\alpha \geq -1$ one may work out the local in time well-posedness for (2.1) in $H_{per}^{3/2+}([0, T])$ via the usual energy method. Without recourse to dispersive effects, the proof is identical to that for the inviscid Burgers equation $u_t + (u^2)_x = 0$. Hence we omit the detail.

With the help of techniques in nonlinear dispersive equations and specific features of the equation, then, the global in time well-posedness for (2.1) may be established in $H_{per}^{-1/2+}([0, T])$ in the case of $\alpha = 2$, namely the KdV equation (see [CKS⁺03]), and in $H_{per}^{0+}([0, T])$ in the case of $\alpha = 1$, the BO equation (see [Mol08]), respectively. For non-integer values of α , on the other hand, the existence matter for (2.1) seems not properly understood in spaces of low regularity. In the non-periodic setting, the global well-posedness in the energy space was recently settled in [KMR12] for (2.1), where $1 < \alpha < 2$ and u^{p+1} is in place of u^2 , $p \leq 2\alpha$, but the proof seems to break down in the periodic setting.

In what follows we will work in an appropriate subspace, say, X of $H_{per}^{\alpha/2}([0, T])$, where the initial value problem associated with (2.1) is well-posed for some interval of time and where $H, P, M : X \rightarrow \mathbb{R}$ are smooth.

A periodic traveling wave of (2.1) takes the form $u(x, t) = u(x - ct - x_0)$, where $c \in \mathbb{R}$ represents the wave speed, $x_0 \in \mathbb{R}$ is the spatial translate and u is T -periodic, satisfying after integration that

$$(2.7) \quad \Lambda^\alpha u - u^2 + cu + a = 0$$

for some $a \in \mathbb{R}$ (in the sense of distributions). Alternatively it arises as a critical point of

$$(2.8) \quad E(u; c, a) = H(u) + cP(u) + aM(u).$$

Indeed it is straightforward to show from (2.2) and (2.3), (2.4) (see (2.5)) that (2.7) is equivalent to

$$(2.9) \quad \delta E(u; c, a) = 0.$$

Henceforth we will write a periodic traveling wave of (2.1) as $u = u(\cdot; c, a)$. In a more comprehensive description, it is specified by four parameters c , a and T , x_0 (see Section 6). But $T > 0$ is fixed. Moreover, corresponding to the translational symmetry of the equation, x_0 is inconsequential in the present development. Hence we may mod it out.

In the present notation, a solitary wave, whose profile vanishes at infinity, corresponds to $a = 0$ and $T = \infty$ simultaneously in (2.7).

In the case of $\alpha = 2$, periodic traveling waves of (2.1), namely the KdV equation, are well known to be expressible in a closed form, involving Jacobi's elliptic functions (see [KdV95], for example). In the case of $\alpha = 1$, moreover, Benjamin [Ben67] manipulated the Poisson summation formula and explicitly calculated periodic traveling waves of (2.1). In general, the existence of periodic traveling waves of (1.1), for a broad range of dispersion operators and nonlinearities including (2.1), may follow from variational arguments, although one may lose the explicit form of the solution. In the $H^{\alpha/2}$ -subcritical case, i.e. $\alpha > 1/3$, in particular, a family of periodic traveling waves of (2.1) arises as energy minimizers constrained to conservations of the momentum and the mass, generalizing ground states in the solitary wave setting.

Proposition 2.2 (Existence, symmetry and regularity). *Let $1/3 < \alpha \leq 2$ be fixed. A minimizer u for H subject to that P and M are conserved exists in $H_{per}^{\alpha/2}([0, T])$ for each $0 < T < +\infty$ and it satisfies (2.7) for some $c \neq 0$ and $a \in \mathbb{R}$. Furthermore u depends upon c and a in the C^1 manner.*

Moreover $u(\cdot; c, a)$ may be chosen to be even and strictly decreasing over the interval $[0, T/2]$, and $u(\cdot; c, a) \in H_{per}^\infty([0, T])$.

Proof. We claim that it suffices to consider $a = 0$ in (2.7). For, otherwise, we claim that we may assume without loss of generality that c and M are of opposite sign and $a \geq 0$. Indeed, in the case when c and M are of the same sign, notice that (2.1) is time reversible and performing the variable change $t \mapsto -t$ in (2.1) switches the sign of c in (2.7) while leaving other components of the equation invariant. Once we accomplish that c and M are of opposite sign, we integrate (2.7) over one period and obtain that

$$(2.10) \quad -2P + cM + aT = 0.$$

Since $P \geq 0$ and $T > 0$, by definition, this proves that $a \geq 0$. We then devise the change of variables $u \mapsto u + \frac{1}{2}(\sqrt{c^2 + 4a} - c)$ and rewrite (2.7) as

$$(2.11) \quad \Lambda^\alpha u - u^2 + \gamma u = 0, \quad \text{where } \gamma = \sqrt{c^2 + 4a} > 0.$$

Incidentally this is reminiscent of that (2.1) enjoys the Galilean symmetry under $u(x, t) \mapsto u(x - \lambda t, t) + \lambda$ for any $\lambda \in \mathbb{R}$. To recapitulate, it suffices to take $a = 0$ (and $c > 0$) in (2.7). Accordingly we seek a minimizer for H subject to constant P . (But we shall not a priori assume that $a = 0$ in the stability analysis in Section 4.)

Since $H_{per}^{\alpha/2}([0, T])$ in the range $\alpha > 1/3$ is compactly embedded in $L_{per}^3([0, T])$ by a Sobolev inequality, it is standard from techniques in calculus of variations that

the minimization problem

$$E_1 = \inf \{H(u) + P(u) : u \in H_{per}^{\alpha/2}([0, T])\}$$

is attained, say, at $u_1 \in H_{per}^{\alpha/2}([0, T])$. The proof is elementary. Hence we omit the detail, but we merely pause to remark that

$$H(u) + P(u) = \frac{1}{2} \int_0^T (u^2 + u\Lambda^\alpha u) dx - \frac{1}{3} \int_0^T u^3 dx$$

is weakly lower semi-continuous with respect to the $H_{per}^{\alpha/2}([0, T])$ norm. Furthermore u_1 satisfies (2.11) with $\gamma = 1$, the associated Euler-Lagrange equation, in the sense of distributions.

Thanks to the scaling invariance (2.6), then, the constrained minimization problem with parameter (abusing notation) $P > 0$,

$$E = \inf \{H(u) : u \in H_{per}^{\alpha/2}([0, T]), P(u) = P\}$$

is attained at $u \in H_{per}^{\alpha/2}([0, T])$ if and only if $u(x) = \lambda^{\alpha/2} u_1(\lambda x)$ for some constant $\lambda > 0$ chosen to ensure that $P(u) = P$. Moreover it is immediate that u satisfies (2.11) for some $\gamma > 0$, serving as the Lagrange multiplier associated with the momentum constraint, in the sense of distributions. The existence assertion therefore follows. Furthermore it is standard from calculus of variations that u depends upon γ and, in turn, c and a in the C^1 manner.

To proceed, since the symmetric-decreasing rearrangement of u strictly decreases $\int_0^T u\Lambda^\alpha u dx$ for $0 < \alpha \leq 2$ while leaving $\int_0^T u^3 dx$ invariant, it follows from standard rearrangement arguments that the minimizer for H subject to conservations of P and M must symmetrically decrease away from a point of principal elevation. The symmetry and monotonicity assertion therefore follows by virtue of the translational invariance of (2.7). (Notice that unlike in the solitary waves setting, for which $a = 0$ and $T = \infty$ simultaneously, a periodic energy minimizer under constraints need not be positive everywhere.)

It remains to address the smoothness of a periodic solution of (2.7), or equivalently,

$$(2.12) \quad u = (\Lambda^\alpha + 1)^{-1} u^2$$

after reduction to $a = 0$ and $c = 1$ and after inversion; the validity of (2.12) will be specified in the course the proof. We first claim that if $u \in H_{per}^{\alpha/2}([0, T])$ satisfies (2.12) then $u \in L_{per}^\infty([0, T])$. In the case of $\alpha > 1$ this follows immediately from a Sobolev inequality, while in the case of $1/3 < \alpha \leq 1$ a proof based upon bounds for the resolvent $(\Lambda^\alpha + 1)^{-1}$ is found in [FL12, Lemma A.3], for example, albeit in the solitary wave setting. Specifically $\widehat{\frac{1}{|\xi|^{\alpha+1}}} \in L_{per}^r([0, T])$ for $0 < \alpha < 1$ and for $r > \frac{1}{1-\alpha}$, whence $u \in L_{per}^\infty([0, T])$ after iterating (2.12) sufficiently many times.

We then promote $u \in H_{per}^{\alpha/2}([0, T]) \cap L_{per}^\infty([0, T])$ to $H_{per}^\alpha([0, T])$, since the Plancherel theorem and Young's inequality in the Fourier space manifest that

$$\begin{aligned} \|\Lambda^\alpha u\|_{L^2} &= \left\| \frac{\Lambda^\alpha}{\Lambda^\alpha + 1} u^2 \right\|_{L^2} = \left\| \frac{|\xi|^\alpha}{|\xi|^\alpha + 1} \hat{u} * \hat{u} \right\|_{L^2} \\ &\leq C \|\hat{u}\|_{L^1} \|\hat{u}\|_{L^2} \leq C \|u\|_{L^\infty} \|u\|_{L^2} < \infty. \end{aligned}$$

Here and elsewhere C means a positive generic constant. Similarly

$$\|\Lambda^{2\alpha}u\|_{L^2} = \left\| \frac{\Lambda^{2\alpha}}{\Lambda^\alpha + 1} u^2 \right\|_{L^2} \leq C \|\hat{u}\|_{L^1} \|\xi|^\alpha \hat{u}\|_{L^2} \leq C \|u\|_{L^\infty} \|\Lambda^\alpha u\|_{L^2} < \infty,$$

and $u \in H_{per}^\infty([0, T])$ follows successively. \square

In light of Proposition 2.2 we may regard H , P , M and in turn E , evaluated at a periodic, constrained energy minimizer $u(\cdot; c, a)$ for (2.1), as C^1 functions of c and a . We are then permitted to differentiate them with respect to c or a . Evaluating (2.9) at a periodic constrained minimizer and differentiating with respect to c and a , for example, we obtain that

$$(2.13) \quad \delta^2 E(u)u_c = -\delta P(u) \quad \text{and} \quad \delta^2 E(u)u_a = -\delta M(u).$$

Moreover

$$(2.14) \quad \begin{aligned} M_c(u(\cdot; c, a)) &= \langle \delta M(u), u_c \rangle = \langle -\delta^2 E(u)u_a, u_c \rangle \\ &= \langle u_a, -\delta^2 E(u)u_c \rangle = \langle u_a, \delta P(u) \rangle = P_a(u(\cdot; c, a)). \end{aligned}$$

Incidentally (2.13) and (2.14) hold for smooth, periodic traveling waves for abstract Hamiltonian systems under certain structural assumptions; details are discussed in [BH12].

Remark 2.3 (Extension to power-law nonlinearities). One may repeat the arguments in the proof of Proposition 2.2 for KdV type equations with general power-law nonlinearities

$$(2.15) \quad u_t - \Lambda^\alpha u_x + (u^{p+1})_x = 0$$

to obtain periodic traveling waves. Here $0 < \alpha \leq 2$ and $0 < p < p_{max}$ is an integer, where

$$(2.16) \quad p_{max} := \begin{cases} \frac{2\alpha}{1-\alpha} & \text{for } \alpha < 1, \\ +\infty & \text{for } \alpha \geq 1. \end{cases}$$

They minimize in $H_{per}^{\alpha/2}([0, T])$ the Hamiltonian

$$\int_0^T \left(\frac{1}{2} u \Lambda^\alpha u - \frac{1}{p+2} u^{p+2} \right) dx$$

constrained to conservations of P and M , defined as in (2.3) and (2.4), respectively. The condition $0 < p < p_{max}$, which is vacuous if $\alpha \geq 1$, ensures that the nonlinearity in (2.15) is $H^{\alpha/2}$ -subcritical and that $H_{per}^{\alpha/2}([0, T]) \subset L_{per}^{p+2}([0, T])$ compactly. In case $p = 1$ it is equivalent to that $\alpha > 1/3$.

Remark 2.4 (Periodic vs. solitary waves). In the non-periodic setting, Weinstein in [Wei87] (see also [FL12]) demonstrated that (2.7) in the range $\alpha > 1/3$ admits a solitary wave, for which $a = 0$ and $T = \infty$, by seeking an optimizer in $H^{\alpha/2}(\mathbb{R})$ for the Gagliardo-Nirenberg-Sobolev inequality

$$(2.17) \quad \left(\int |u|^3 \right)^{2\alpha} \leq C \left(\int u \Lambda^\alpha u \right) \left(\int u^2 \right)^{(\alpha-1)+2\alpha}.$$

If $\alpha > 1/2$ so that (2.7) is L^2 -subcritical, in addition, a solitary wave agrees with a minimizer for the Hamiltonian subject to constant momentum. (In the solitary wave setting the mass constraint does not play a role since $\delta M = 1 \notin L^2(\mathbb{R})$.) In the range $\alpha > 1/2$, then, periodic constrained minimizers for (2.1), constructed via

Proposition 2.2, are expected to tend to the solitary wave as the period increases to infinity. This in some sense generalizes the notion of the homoclinic limit in the classical case of $\alpha = 2$.

If[†] $1/3 < \alpha < 1/2$, on the other hand, constrained energy minimizers, while they exist in the periodic wave setting, are unlikely to achieve a limiting state with bounded energy (the $H^{\alpha/2}$ -norm) at the solitary wave limit. Rather the argument in [Wei87] will lead to a family of periodic traveling waves of (2.1) that each optimizes (2.17) among mean zero functions and which are expected to converge to the solitary wave, obtained in [Wei87], as the period increases to infinity. In the range $\alpha < 1/2$, therefore, a periodic Gagliardo-Nirenberg-Sobolev optimizer must be distinct from the constrained energy minimizer constructed via Proposition 2.2.

One is able to construct periodic traveling waves of (1.1), for a general class of dispersion operators and nonlinearities including (2.1) with $\alpha \geq -1$, at least for small amplitudes, via a perturbative method, e.g. the local bifurcation theory. In the solitary wave setting, in contrast, Pohozaev identities techniques dictate that (2.7) for $\alpha \leq 1/3$ does not admit any nontrivial solutions in $H^{\alpha/2}(\mathbb{R}) \cap L^3(\mathbb{R})$.

3. NONDEGENERACY OF THE LINEARIZATION

Throughout the section we fix a periodic traveling wave $u(\cdot; c, a)$ of (2.1), whose existence follows from Proposition 2.2. This section concerns the nondegeneracy of the linearization associated with (2.7) at such constrained energy minimizer.

Proposition 3.1 (Nondegeneracy). *Let $1/3 < \alpha \leq 2$. If $u(\cdot; c, a) \in H_{per}^{\alpha/2}([0, T])$ for some $c \neq 0$, $a \in \mathbb{R}$ and for some $T > 0$ minimizes H subject to that P and M are conserved then the associated linearized operator*

$$(3.1) \quad \delta^2 E(u; c, a) = \Lambda^\alpha - 2u + c$$

acting on $L_{per}^2([0, T])$ is nondegenerate. That is to say,

$$\ker(\delta^2 E(u; c, a)) = \text{span}\{u_x\}.$$

As mentioned in the introduction, the nondegeneracy of the linearization is of fundamental importance in the stability of traveling waves and the blowup analysis for the related time evolution equation; see [Wei87, Lin08, KMR12] among others. But to establish the property is far from being trivial, though. For example, one may cook up a polynomial nonlinearity, say, $f(u)$ so that the kernel of $-\partial_x^2 + f'(u)$ at a periodic traveling wave u is two dimensional at isolated points. The nondegeneracy of the linearization is usually imposed in terms of a spectral assumption, although it may be proven in few special cases.

For (local) generalized KdV equations, the nondegeneracy at a periodic traveling wave was shown in [Joh09] to be equivalent to that the wave amplitude not be a critical point of the period, with the help of the Sturm-Liouville oscillation theorems for ODEs, and it was likewise verified in [Kwo89], for example, at ground states. Amick and Toland in [AT91] demonstrated the property for the BO equation, both in the periodic and solitary wave settings, by relating the nonlocal traveling wave equation to a fully nonlinear ODE via complex analysis techniques; unfortunately,

[†] The stability for the L^2 -critical equation, i.e. $\alpha = 1/2$, is rather delicate and it is outside the scope of the present development. Instead we refer the reader to [KMR12], for example.

their arguments are extremely specific to the BO equation. Angulo Pava and Natali in [APN08] made an alternative proof based upon the theory of totally positive operators, but it in essence necessitates an explicit form of the solution. A satisfactory understanding of the nondegeneracy thus seems largely missing for nonlocal equations. The main obstruction is that shooting arguments and other ODE techniques, which seem crucially important for local equations, may not be applicable to nonlocal operators.

Nevertheless, Frank and Lenzmann in their recent work [FL12] successfully settled the nondegeneracy at ground states for a family of nonlinear nonlocal equations with fractional derivatives. Their idea lies in to find a suitable substitute for the Sturm-Liouville theory to count the number of sign changes in eigenfunctions for a fractional Laplacian with potential. Our proof of Proposition 3.1 follows along the same line as the arguments in [FL12, Section 3], but with appropriate modifications to accommodate the periodic nature of the problem.

Lemma 3.2 (Oscillation of eigenfunctions). *Under the hypothesis of Proposition 3.1 an eigenfunction in $H_{per}^{\alpha/2}([0, T]) \cap C_{per}^0([0, T])$ corresponding to the j -th eigenvalue of $\delta^2 E(u)$ changes its sign at most j times over the periodic interval $[0, T]$.*

The regularity assertion of eigenfunctions follows from the last part of Proposition 2.2; see also the discussion in [FL12]. A thorough proof of Lemma 3.2 is found in [FL12] in the solitary wave setting and in [BK04] for $\alpha = 1$. Here we merely hit the main points.

Note that Λ^α , $0 < \alpha < 2$, may be viewed as the Dirichlet-to-Neumann operator for an appropriate (local) elliptic problem set in the periodic half strip $[0, T]_{per} \times [0, \infty)$. Specifically (see [RS12, Theorem 1.1], for example)

$$C(\alpha)\Lambda^\alpha u = \lim_{y \rightarrow 0+} y^{1-\alpha} w_y(\cdot, y),$$

where w solves the (degenerate unless $\alpha = 1$) elliptic boundary value problem

$$\Delta w + \frac{1-\alpha}{y} w_y = 0 \quad \text{in } [0, T]_{per} \times (0, \infty), \quad w = u \quad \text{on } [0, T]_{per} \times \{0\}$$

and $C(\alpha)$ is an explicit constant. Accordingly one derives a variational characterization of (eigenvalues and) eigenfunctions of (3.1) in terms of the Dirichlet type functional

$$\iint_{[0, T]_{per} \times (0, \infty)} |\nabla w(x, y)|^2 y^{1-\alpha} dx dy + \int_0^T (-2u(x) + c) |w(x, 0)|^2 dx$$

in a suitable function class. The assertion then follows from nodal domain bounds a la Courant.

Next, we gather some facts about $\delta^2 E$.

Lemma 3.3 (Properties of $\delta^2 E$). *Under the hypothesis of Proposition 3.1 the followings hold:*

- (L1) $u_x \in \ker(\delta^2 E(u))$; moreover it corresponds to the lowest eigenvalue of $\delta^2 E(u)$ restricted to the sector of odd functions in $L_{per}^2([0, T])$;
- (L2) $1 \leq n_-(\delta^2 E(u)) \leq 2$, where $n_-(\delta^2 E)$ means the number of negative eigenvalues of $\delta^2 E(u)$ acting on $L_{per}^2([0, T])$;
- (L3) $1, u \in \text{range}(\delta^2 E(u))$.

Proof. Differentiating (2.7) with respect to x implies that $\delta^2 E(u)u_x = 0$. Moreover Proposition 2.2 ensures that u may be chosen to satisfy $u_x(x) < 0$ for $0 < x < T/2$. On the other hand, the lowest eigenvalue of $\delta^2 E(u)$ acting on $L^2_{per,odd}([0, T])$ must be simple and the corresponding eigenfunction must be strictly positive over the interval $[0, T/2]$; the proof of such a Perron-Frobenius property is elementary and hence we omit the detail. Therefore zero is the lowest eigenvalue of $\delta^2 E(u)$ restricted to $L^2_{per,odd}([0, T])$ and u_x is a corresponding eigenfunction.

Next, recall that u_x possesses two roots over the periodic interval $[0, T]$ and it belongs to the kernel of $\delta^2 E(u)$. Since it is readily verified from Perron-Frobenius arguments (see, for example, [RS78]) that the eigenfunction associated to the smallest eigenvalue of $\delta^2 E(u)$ can be chosen to be strictly positive, the operator $\delta^2 E(u)$, acting on $L^2_{per}([0, T])$, has at least one negative eigenvalue.

Furthermore, since u is a minimizer for H , and hence E , constrained to conservations of P and M , necessarily,

$$(3.2) \quad \delta^2 E(u)|_{\{\delta P(u), \delta M(u)\}^\perp} \geq 0.$$

This implies by Courant's mini-max principle that $\delta^2 E(u)$ has at most two negative directions, asserting (L2).

Lastly, (2.13) and (2.5) verify (L3). \square

Remark 3.4 (Number of negative directions). Unlike in the solitary wave setting, where $n_-(\delta^2 E) = 1$ at a ground state, $\delta^2 E$ may have up to two negative eigenvalues at a periodic constrained minimizer, characterized as

$$(3.3) \quad \begin{aligned} n_-(\delta^2 E(u; c, a)) &= n_- \begin{pmatrix} M_a(u(\cdot; c, a)) & P_a(u(\cdot; c, a)) \\ M_c(u(\cdot; c, a)) & P_c(u(\cdot; c, a)) \end{pmatrix} \\ &= \# \text{ of sign changes in } 1, M_a, M_a P_c - M_c P_a. \end{aligned}$$

The proof is found in [BH12], for example.

With the above preparations, we now prove the main result of this section.

Proof of Proposition 3.1. Consider the orthogonal decomposition

$$L^2_{per}([0, T]) = L^2_{per,odd}([0, T]) \oplus L^2_{per,even}([0, T]).$$

Since u may be chosen to be even thanks to Proposition 2.2, we note that $L^2_{per,odd}([0, T])$ and $L^2_{per,even}([0, T])$ are invariant subspaces of $\delta^2 E(u)$. Moreover (L1) of Lemma 3.3 implies that

$$\ker(\delta^2 E(u)|_{L^2_{per,odd}([0, T])}) = \text{span}\{u_x\}.$$

It remains to show that $\ker(\delta^2 E(u)|_{L^2_{per,even}([0, T])}) = \{0\}$.

Suppose on the contrary that there were a non-trivial function $\phi \in L^2_{per,even}([0, T])$ such that $\delta^2 E(u)\phi = 0$. Since $\delta^2 E$ has at most two negative eigenvalues by (L2) of Lemma 3.3, such ϕ changes its sign at most twice over the periodic interval $[0, T]$ by Lemma 3.2. Consequently ϕ can be chosen to be positive throughout $[0, T]$, or else there exists $T_1 \in (0, T/2)$ such that ϕ is positive for $|x| < T_1$ and negative for $x \in (-T/2, T_1) \cup (T_1, T/2)$. Since ϕ is in the kernel of $\delta^2 E(u)$, on the other hand, it must be orthogonal to $\text{range}(\delta^2 E(u))$, and in turn to the subspace $\text{span}\{1, u\}$ by (L3) of Lemma 3.3. In particular

$$\langle \phi, \sigma u - \tau \rangle = 0 \quad \text{for all } \sigma, \tau \in \mathbb{R}.$$

Taking $\sigma = 0$ and $\tau = -1$ we see that ϕ cannot be positive throughout $[0, T]$. In case ϕ changes signs at $x = \pm T_1$, noting that u is symmetrically decreasing away from the origin over the interval $(-T/2, T/2)$, we may choose $\tau \in \mathbb{R}$ so that $u - \tau$ is positive in $(-T_1, T_1)$ and negative in $(-T/2, -T_1) \cup (T_1, T/2)$, implying that ϕ cannot be orthogonal to $\{1, u\}$. A contradiction therefore asserts that the kernel of $\delta^2 E$ must consist merely of u_x . \square

One may repeat the argument in the proof of Proposition 3.1 *mutatis mutandis* to establish the nondegeneracy at a periodic, constrained energy minimizer for (2.15) for $0 < \alpha \leq 2$ and $0 < p < p_{max}$, where p_{max} is in (2.16).

4. STABILITY OF CONSTRAINED ENERGY MINIMIZERS

We now turn the attention to the stability of a periodic constrained minimizer for (2.1) with respect to period preserving perturbations.

Recall from Section 2 that the initial value problem associated with (2.1) is well-posed in $X \subset H_{per}^{\alpha/2}([0, T])$ for some interval of time, where $H, P, M : X \rightarrow \mathbb{R}$ are smooth. It suffices to take $X = H_{per}^{\beta}([0, T])$, $\beta > 3/2$.

Throughout the section let $1/3 < \alpha \leq 2$, fixed, and let $u_0(\cdot, c_0, a_0) \in H_{per}^{\alpha/2}([0, T])$ minimize H subject to that P and M are conserved for some $c_0 \neq 0$, $a_0 \in \mathbb{R}$ and for some $T > 0$. In light of Proposition 2.2, then, $u_0 \in X$ and it makes a T -periodic, traveling wave solution of (2.1).

Notice that the evolution of (2.1) remains invariant under a one-parameter group of isometries corresponding to spatial translations. This motivates us to define the group orbit of, say, $u \in X$ as

$$\mathcal{O}_u = \{u(\cdot - x_0) : x_0 \in \mathbb{R}\}.$$

Roughly speaking, $u_0(\cdot; c_0, a_0)$ is said *orbitally stable* if a solution of (2.1) remains close to \mathcal{O}_{u_0} under the norm of X for all future times whenever the initial datum is sufficiently close to the group orbit of u_0 under the norm of X . We shall elaborate on this below in Theorem 4.1.

The present account of orbital stability is inspired by the Lyapunov method. Let

$$(4.1) \quad E_0(u) = H(u) + c_0 P(u) + a_0 M(u).$$

Proposition 2.2 says that $\delta E_0(u_0) = 0$. In other words, u_0 is a critical point of E_0 . Proposition 3.1 furthermore guarantees that the kernel of $\delta^2 E_0(u_0)$ is spanned by u_{0x} . Accordingly u_0 is expected to be orbitally stable if E_0 is “convex” at u_0 . As a matter of fact, one easily verifies that if the spectrum of $\delta^2 E_0(u_0)$ except the simple eigenvalue at the origin were positive and bounded away from zero then u_0 would indeed be orbitally stable.

But (L2) of Lemma 3.3 indicates that $\delta^2 E_0(u_0)$ has at least one negative eigenvalue; consequently u_0 is a nondegenerate saddle of E_0 on X . In the solitary wave setting, incidentally, $\delta^2 E_0$ at a ground state admits exactly one negative eigenvalue. In order to control these (at most two) potentially unstable directions, we observe that the evolution under (2.1) does not take place in the entire space X , but rather on a smooth submanifold of co-dimension two, along which the momentum and the mass are conserved. Specifically let

$$(4.2) \quad P_0 = P(u_0(\cdot; c_0, a_0)), \quad M_0 = M(u_0(\cdot; c_0, a_0))$$

and let

$$\Sigma_0 = \{u \in X : P(u) = P_0, M(u) = M_0\}.$$

Then $\mathcal{O}_{u_0} \subset \Sigma_0$ and a solution of (2.1) issued from Σ_0 remains in Σ_0 at all future times. The goal of this section is to establish the “convexity” of E_0 on Σ_0 .

Theorem 4.1 (Orbital stability). *Let $1/3 < \alpha \leq 2$. If $u_0(\cdot; c_0, a_0) \in H_{per}^{\alpha/2}([0, T])$ for some $c_0 \neq 0$, $a_0 \in \mathbb{R}$ and for some $T > 0$ minimizes H subject to that P and M are conserved then for all $\varepsilon > 0$ sufficiently small there exists a constant $C(\varepsilon) > 0$ such that: if $\phi \in X$ and $\|\phi\|_X \leq \varepsilon$ and if $u(\cdot, t)$ is a solution of (2.1) for some interval of time with the initial condition $u(\cdot, 0) = u_0 + \phi \in \Sigma_0$ then $u(\cdot, t)$ may be continued to a solution for all $t > 0$ such that*

$$(4.3) \quad \sup_{t > 0} \inf_{x_0 \in \mathbb{R}} \|u(\cdot, t) - u_0(\cdot - x_0)\|_X \leq C\|\phi\|_X.$$

If, in addition, the matrix

$$(4.4) \quad \begin{pmatrix} M_a(u(\cdot; c, a)) & P_a(u(\cdot; c, a)) \\ M_c(u(\cdot; c, a)) & P_c(u(\cdot; c, a)) \end{pmatrix}$$

is nonsingular at $u_0(\cdot; c_0, a_0)$ then the same conclusion holds for all $\phi \in X$ such that $\|\phi\|_X \leq \varepsilon$.

Theorem 4.1 makes rigorous that a periodic, constrained energy minimizer for (2.1) is orbitally stable with respect to nearby solutions with the same momentum and the same mass as the underlying wave, and with respect to arbitrary nearby solutions provided that the constraint set Σ_0 is nondegenerate, i.e. (4.4) is nonsingular, at the underlying wave.

A solitary[†] wave $u_0(\cdot; c_0)$ of (2.1) (not necessarily a ground state), in comparison, was shown in [GSS87] to be orbitally stable provided that

$$(4.5) \quad \ker(\delta^2 E_0(u_0)) = \text{span}\{u_{0x}\}, \quad n_-(\delta^2 E_0(u_0)) = 1, \quad P_c(u_0(\cdot; c_0)) > 0.$$

(Notice that the assumption in [GSS87] that the symplectic form of a Hamiltonian system be onto is dispensable for the stability; see the remark directly following [GSS87, Theorem 2].) Conditions in (4.5) were further verified in [BSS87] (see also [SS90, Wei87]) to hold if and only if $\alpha > 1/2$. Recall incidentally from Remark 2.4 that in the range $\alpha > 1/2$ a solitary wave arises as an energy minimizer subject to constant momentum. Accordingly Theorem 4.1 may be regarded as extending the well-known result about solitary waves to periodic traveling waves.

A natural approach toward Theorem 4.1 is to repeat the arguments in the proof in [GSS87] and derive a stability criterion analogous to (4.5); see [Joh09], for example, for (local) generalized KdV equations, where the last condition in (4.5) was suitably modified. But it is difficult to exactly count $n_-(\delta^2 E_0)$ in the periodic wave setting, though (see Remark 3.4). By exploiting that the underlying wave arises as a constrained energy minimizer, instead, and utilizing merely the nondegeneracy of the linearization, our proof of Theorem 4.1 needs not information about the number of negative eigenvalues of $\delta^2 E_0$, other than the upper bound in Lemma 3.3.

In the range $\alpha > 1/2$ recall from Remark 2.4 that periodic, constrained energy minimizers for (2.1), which are orbitally stable by Theorem 4.1, are expected to tend

[†] A solitary wave corresponds to $a = 0$ and $T = \infty$ in (2.7). Hence it depends, up to spatial translations, merely upon the wave speed.

to the solitary wave as the period increases to infinity, and the limiting solitary wave is orbitally stable as well (see [BSS87], for example). In the range $1/3 < \alpha < 1/2$, Theorem 4.1 indicates that orbitally stable, constrained energy minimizers for (2.1) exist in the periodic wave setting, but they are unlikely attain a limiting wave form with finite energy at the solitary wave limit. In light of Remark 2.4, rather, periodic Gagliardo-Nirenberg-Sobolev (see (2.17)) optimizers, which are expected to be unstable at least for sufficiently large periods, will converge to the unstable solitary wave as the period increases to infinity.

The proof of Theorem 4.1 relies upon the coercivity of E_0 on Σ_0 in a neighborhood of the group orbit of u_0 . In what follows we introduce the semidistance $\rho : X \rightarrow \mathbb{R}$, defined by

$$\rho(u, v) = \inf_{x_0 \in \mathbb{R}} \|u - v(\cdot - x_0)\|_X.$$

We may then rewrite (4.3) as $\sup_{t > 0} \rho(u(\cdot, t), u_0) \leq C \|u(\cdot, 0) - u_0\|_X$.

Lemma 4.2 (Coercivity). *Under the hypothesis of Theorem 4.1 there exist $\varepsilon > 0$ and $C(\varepsilon) > 0$ such that if $u \in \Sigma_0$ with $\rho(u, u_0) < \varepsilon$ then*

$$(4.6) \quad E_0(u) - E_0(u_0) \geq C \rho(u, u_0)^2.$$

Proof. The proof closely resembles that of [GSS87, Theorem 3.4] or [Joh09, Proposition 4.3]. Here we include the detail for completeness.

Throughout the proof and the following, C denotes a positive generic constant; C which appears in different places in the text needs not be the same.

The implicit function theorem (see [BSS87, Lemma 4.1], for example) implies that for $\varepsilon > 0$ sufficiently small and for an ε -neighborhood $\mathcal{U}_\varepsilon := \{u \in X : \rho(u, u_0) < \varepsilon\}$ of \mathcal{O}_{u_0} there exists a unique C^1 map $\omega : \mathcal{U}_\varepsilon \rightarrow \mathbb{R}$ such that

$$\omega(u_0) = 0 \quad \text{and} \quad \langle u(\cdot + \omega(u)), u_{0x} \rangle = 0$$

for all $u \in \mathcal{U}_\varepsilon$. Since E_0 is invariant under spatial translations, it suffices to prove (4.6) along the translates $u(\cdot + \omega(u))$.

Since u_0 minimizes H , and hence E_0 , constrained to that $P = P_0$ and $M = M_0$, necessarily,

$$(4.7) \quad \delta^2 E_0(u_0)|_{\mathcal{T}_0} \geq 0, \quad \text{where } \mathcal{T}_0 = \{\delta P(u_0), \delta M(u_0)\}^\perp$$

denotes the tangent space in X to the sub-manifold Σ_0 at u_0 . We then fix $u \in \mathcal{U}_\varepsilon \cap \Sigma_0$ and write

$$(4.8) \quad u(\cdot + \omega(u)) = u_0 + C_1 \delta P(u_0) + \left(C_2 - C_1 \frac{\langle \delta M(u_0), \delta P(u_0) \rangle}{\langle \delta M(u_0), \delta M(u_0) \rangle} \right) \delta M(u_0) + y,$$

where $C_1, C_2 \in \mathbb{R}$ and $y \in (\text{span}\{u_0\} \cup \mathcal{T}_0) \cap \{u_{0x}\}^\perp$. Note that $C_1 = C_2 = y = 0$ at $u = u_0$.

Let $\phi = u(\cdot + \omega(u)) - u_0$ and note we may assume that $\|\phi\|_X < \varepsilon$ possibly after replacing u_0 by $u_0(\cdot - x_0)$ for some $x_0 \in \mathbb{R}$. Since P and M remain invariant under spatial translations, Taylor's theorem then manifests that

$$(4.9) \quad \begin{aligned} P(u) &= P(u(\cdot + \omega(u))) = P(u_0) + \langle \delta P(u_0), \phi \rangle + O(\|\phi\|_X^2), \\ M(u) &= M(u(\cdot + \omega(u))) = M(u_0) + \langle \delta M(u_0), \phi \rangle + O(\|\phi\|_X^2). \end{aligned}$$

Since $\langle \delta M(u_0), \phi \rangle = C_2 \langle \delta M(u_0), \delta M(u_0) \rangle = C_2 T$ by (4.8) and (2.5) we infer from the latter equation in (4.9) that $C_2 = O(\|\phi\|_X^2)$. Similarly, since

$$\begin{aligned} \langle \delta P(u_0), \phi \rangle &= C_1 \left(\langle \delta P(u_0), \delta P(u_0) \rangle - \frac{\langle \delta M(u_0), \delta P(u_0) \rangle^2}{\langle \delta M(u_0), \delta M(u_0) \rangle} \right) + C_2 \langle \delta P(u_0), \delta M(u_0) \rangle \\ &= C_1 \left(\|u_0\|_{L^2_{per}(0,T]}^2 - \frac{M_0^2}{T} \right) - C_2 M_0, \end{aligned}$$

the Cauchy-Schwarz inequality together with the former equation in (4.9) and (2.5) yields that $C_1 = O(\|\phi\|_X^2)$.

Furthermore, the invariance of E_0 under spatial translations and Taylor's theorem lead to

$$E_0(u) = E_0(u(\cdot + \omega(u))) = E_0(u_0) + \frac{1}{2} \langle \delta^2 E_0(u_0) \phi, \phi \rangle + o(\|\phi\|_X^2).$$

Using (4.8) and $C_1, C_2 = O(\|\phi\|_X^2)$ we then find that

$$E_0(u) - E_0(u_0) = \frac{1}{2} \langle \delta^2 E_0(u_0) \phi, \phi \rangle + o(\|\phi\|_X^2) = \frac{1}{2} \langle \delta^2 E_0(u_0) y, y \rangle + O(\|\phi\|_X^2).$$

Since $\ker(\delta^2 E_0(u_0)) = \text{span}\{u_{0x}\}$ by Proposition 3.1, moreover, it follows that

$$y \in \mathcal{T}_0 \cap \ker(\delta^2 E_0(u_0))^\perp,$$

whence using (4.7) we obtain that $\langle \delta^2 E_0(u_0) y, y \rangle \geq C \|y\|_X^2$. Finally a straightforward calculation reveals that

$$\begin{aligned} \|y\|_X &\geq \left\| \|\phi\|_X - \left\| C_1 \delta P(u_0) + \left(C_2 - C_1 \frac{\langle \delta M(u_0), \delta P(u_0) \rangle}{\langle \delta M(u_0), \delta M(u_0) \rangle} \right) \delta M(u_0) \right\|_X \right\| \\ &\geq \|\phi\|_X - C \|\phi\|_X^2 \end{aligned}$$

so that

$$E_0(u) - E_0(u_0) \geq C \|\phi\|_X^2 = C \|u(\cdot + \omega(u)) - u_0\|_X^2 \geq C \rho(u, u_0)^2,$$

as claimed. \square

Proof of Theorem 4.1. The proof is similar to that of [Joh09, Lemma 4.1] for (local) generalized KdV equations.

Let $\varepsilon_0 > 0$ be such that Lemma 4.2 holds and let $\phi \in X$ satisfy $\rho(u_0 + \phi, u_0) \leq \varepsilon$ for some $0 < \varepsilon < \varepsilon_0$ sufficiently small. By replacing ϕ by $\phi(\cdot - x_0)$ for some $x_0 \in \mathbb{R}$, if necessary, we may assume without loss of generality that $\|\phi\|_X \leq \varepsilon$. Since u_0 is a critical point of E_0 , then, Taylor's theorem implies that $E_0(u_0 + \phi) - E_0(u_0) \leq C\varepsilon^2$. Furthermore, notice that if $u_0 + \phi \in \Sigma_0$ then the unique solution $u(\cdot, t)$ of (2.1) with the initial condition $u(\cdot, 0) = u_0 + \phi$ must lie in Σ_0 as long as the solution exists. Since $E_0(u(\cdot, t)) = E_0(u(\cdot, 0)) = E_0(u_0 + \phi)$ independently of t , on the other hand, Lemma 4.2 implies that $\rho(u(\cdot, t), u_0)^2 \leq C\varepsilon^2$ for all $t \geq 0$. This verifies the first part of Theorem 4.1.

In the case where $u_0 + \phi$ is not required to be in Σ_0 , but that the matrix (4.4) is nonsingular at $u_0(c_0, a_0)$, we utilize the nondegeneracy of the constraint set, i.e. that the mapping

$$(c, a) \mapsto (P(u(\cdot; c, a)), M(u(\cdot; c, a)))$$

is a period-preserving diffeomorphism from a neighborhood of (c_0, a_0) onto a neighborhood of (P_0, M_0) . We may therefore find $c, a \in \mathbb{R}$ such that $|c| + |a| = O(\varepsilon)$ and $u_\varepsilon(\cdot; c_0 + c, a_0 + a)$ is a T -periodic traveling wave solution of (2.1) satisfying

$$P(u_\varepsilon(\cdot; c_0 + c, a_0 + a)) = P(u_0 + \phi) \quad \text{and} \quad M(u_\varepsilon(\cdot; c_0 + c, a_0 + a)) = M(u_0 + \phi).$$

Let

$$E_\varepsilon(u) = E_0(u) + cP(u) + aM(u).$$

We may furthermore assume that u_ε minimizes E_ε subject to that P and M are conserved. Accordingly we rerun the argument in the proof of Lemma 4.2 and show that

$$E_\varepsilon(u) - E_\varepsilon(u_\varepsilon) \geq C\rho(u, u_\varepsilon)^2$$

so long as $\rho(u, u_\varepsilon)$ is sufficiently small. Since u_ε is a critical point of E_ε , moreover, $E_\varepsilon(u(\cdot, t)) - E_\varepsilon(u_\varepsilon) = E_\varepsilon(u_0 + \phi) - E_\varepsilon(u_\varepsilon) \leq C\varepsilon^2$ for all $t \geq 0$. By the triangle inequality we then find that

$$\begin{aligned} \rho(u(\cdot, t), u_0)^2 &\leq C(\rho(u(\cdot, t), u_\varepsilon)^2 + \rho(u_\varepsilon, u_0)^2) \\ &\leq C(E_\varepsilon(u(\cdot, t)) - E_\varepsilon(u_\varepsilon)) + \|u_\varepsilon - u_0\|_X \leq C\varepsilon^2 \end{aligned}$$

for all $t \geq 0$. In other words, $u_0(\cdot; c_0, a_0)$ is orbitally stable to small perturbations that “slightly” change P and M . \square

One may repeat the argument in the proof of Theorem 4.1 *mutatis mutandis* to establish the orbital stability of a periodic, constrained energy minimizer for (2.15) for $0 < \alpha \leq 2$ and $0 < p < p_{max}$, where p_{max} is in (2.16).

5. ADAPTATION TO EQUATIONS OF REGULARIZED LONG WAVE TYPE

The results in Section 2 through Section 4 may readily be adapted to other nonlinear dispersive equations with fractional dispersion. To illustrate this, we shall discuss equations of regularized long wave (RLW) type

$$(5.1) \quad u_t - u_x + \Lambda^\alpha u_t + (u^2)_x = 0,$$

where $0 < \alpha \leq 2$. Here $t \in \mathbb{R}$ denotes the temporal variable and $x \in \mathbb{R}$ is the spatial variable, $u = u(x, t)$ is real-valued. Recall that Λ is defined via its multiplier $|\xi|$ in the Fourier space.

In the case of $\alpha = 2$, notably, (5.1) recovers the Benjamin-Bona-Mahony (BBM) equation, which was advocated in [BBM72] as an alternative to the KdV equation; in particular solutions of the initial value problem for the BBM equation were argued to enjoy better smoothness than those for the KdV equation and it was named the regularized long wave equation. For other values of α , likewise, (5.1) “regularizes” its KdV counterpart in (2.1). In a certain, weakly nonlinear and long wavelengths regime, where $u_x + u_t = o(1)$ and both the KdV and the BBM equations usually serve as valid models, (5.1) is formally equivalent to (2.1).

The present treatment may extend *mutatis mutandis* to general power-law nonlinearities (see Remark 2.3), but we choose to work with the quadratic nonlinearity to simplify the exposition.

Related stability analysis in the case of $\alpha = 1, 2$ is found, for example, in [ASB11] and [Joh10], respectively.

In the range $\alpha \geq 0$ one may work out the local in time well-posedness for (5.1) in $H_{per}^\beta([0, T])$, where $\beta > \max(0, (3 - \alpha)/2)$, via the usual energy method, corroborating that (5.1) regularizes (2.1) (see Remark 2.1). The proof is straightforward. Hence we omit the detail. On account of the smoothing effect of $(1 - \partial_x^2)^{-1}$, then, the global in time well-posedness for (5.1) may be settled in $H^\beta(\mathbb{R})$, $\beta \geq 0$, in the case of $\alpha = 2$, the BBM equation (see [BT09], for example). In the periodic setting, on the other hand, the existence matter for (5.1) seems not properly understood in spaces of low regularity even in the classical limiting case. For the present purpose, however, an adequate short-time well-posedness theory suffices.

Throughout the section we'll work in an appropriate subspace (abusing notation) X of $L_{per}^2([0, T])$, where the initial value problem for (5.1) is well-posed at least for some interval of time and where $T > 0$ is fixed.

Notice that (5.1) possesses three conserved quantities (abusing notation)

$$(5.2) \quad H(u) = \int_0^T \left(\frac{1}{2}u^2 - \frac{1}{3}u^3 \right) dx$$

and

$$(5.3) \quad P(u) = \int_0^T \frac{1}{2}u(1 + \Lambda^\alpha)u \, dx,$$

$$(5.4) \quad M(u) = \int_0^T (1 + \Lambda^\alpha)u \, dx,$$

which correspond to the Hamiltonian and the momentum, the mass, respectively. Throughout the section we will use H and P, M for those in (5.2) and (5.3), (5.4), respectively. It is readily verified that $H, P, M : X \rightarrow \mathbb{R}$ are smooth and translationally invariant. Notice that (5.1) is written in the Hamiltonian form as

$$u_t = J\delta H(u),$$

where $J = (1 + \Lambda^\alpha)^{-1}\partial_x$ and recall that δ denotes variational differentiation.

We promptly seek a periodic traveling wave of (5.1) of the form $u(x, t) = u(x - ct - x_0)$, where $c \in \mathbb{R}$, $x_0 \in \mathbb{R}$ and u is T -periodic, satisfying by quadrature that

$$(5.5) \quad c(1 + \Lambda^\alpha)u + u - u^2 + a = 0$$

for some $a \in \mathbb{R}$ in the sense of distributions. Equivalently it satisfies (again, abusing notation) that

$$(5.6) \quad \delta E(u; c, a) := \delta(H(u) + cP(u) + aM(u)) = 0.$$

We write a periodic traveling wave of (5.1) as $u = u(\cdot; c, a)$ with the understanding that $T > 0$ is arbitrary but fixed and we may mod out $x_0 \in \mathbb{R}$.

Below we record the existence, symmetry, and regularity properties for a family of periodic traveling waves of (5.1) which arise as energy minimizers subject to that the momentum and the mass are conserved.

Lemma 5.1 (Existence of RLW constrained energy minimizers). *Let $1/3 < \alpha \leq 2$. A minimizer u for H , defined in (5.2), constrained to conservations of P and M , defined in (5.3) and (5.4), respectively, exists in $H_{per}^{\alpha/2}([0, T])$ for each $0 < T < +\infty$ and it satisfies (5.5) for some $c \neq 0$ and $a \in \mathbb{R}$. Furthermore u depends upon c and a in the C^1 manner.*

Moreover $u(\cdot; c, a)$ may be chosen to be even and strictly decreasing over the interval $[0, T/2]$, and $u(\cdot; c, a) \in H_{per}^\infty([0, T])$.

Proof. The proof is nearly identical to that of Proposition 2.2. Indeed if $u(x; c, a)$ is T -periodic and satisfies (5.5) for some $c \neq 0$ and $a \in \mathbb{R}$ then

$$(5.7) \quad (c+1)u\left(\left(\frac{c+1}{c}\right)^{1/\alpha}x; 1, c^2a\right)$$

is $\left(\frac{c+1}{c}\right)^{1/\alpha} T$ -periodic and satisfies (5.5) with $c = 1$. Therefore we may take $c = 1$ in (5.5), which brings us to (2.7) with $c = 2$. We omit the remaining detail. \square

We demonstrate the nondegeneracy of the linearization associated with (5.6) at a periodic, constrained energy minimizer for (5.1).

Lemma 5.2 (Nondegeneracy). *Let $1/3 < \alpha \leq 2$. If $u(\cdot; c, a) \in H_{per}^{\alpha/2}([0, T])$ for some $c \neq 0$, $a \in \mathbb{R}$ and for some $T > 0$ minimizes H constrained to conservations of P and M then*

$$\ker(\delta^2 E(u; c, a)) = \text{span}\{u_x\},$$

where

$$(5.8) \quad \delta^2 E(u; c, a) = c(1 + \Lambda^\alpha) + 1 - 2u$$

is an operator acting on $L_{per}^2([0, T])$.

Proof. The proof is nearly identical to that of Proposition 3.1. Referring to [FL12, Section 3], Lemma 3.2 holds for (5.8). Moreover (L1) and (L2) of Lemma 3.3 hold, and the main difference from the proof of Proposition 3.1 lies in that

$$\delta P(u) = (1 + \Lambda^\alpha)u \quad \text{and} \quad \delta M(u) = 1$$

in place of (2.5).

Evaluating (5.6) at a constrained energy minimizer and differentiating with respect to c and a , we obtain that (see (2.13))

$$\delta^2 E(u)u_c = -\delta P(u) \quad \text{and} \quad \delta^2 E(u)u_a = -\delta M(u),$$

respectively. Therefore $1, (1 + \Lambda^\alpha)u \in \text{range}(\delta^2 E(u))$. Unlike $\delta P(u) = u$ in the KdV setting, however, $(1 + \Lambda^\alpha)u$ may not be strictly monotone over $[0, T/2]$. Appealing to (5.5), instead, we find that

$$c(1 + \Lambda^\alpha)u = u^2 - u - a,$$

which implies that $u^2 - u \in \text{range}(\delta^2 E)$. Moreover a direct calculation reveals that

$$\delta^2 E(u)u = c(1 + \Lambda^\alpha)u + u - 2u^2 = -u^2 - a.$$

Therefore $u \in \text{range}(\delta^2 E)$, asserting (L3) of Lemma 3.3. The proof is then identical to Proposition 3.1. Hence we omit the remaining detail. \square

Repeating the arguments in the proof of Theorem 4.1, ultimately, we establish the orbital stability of a periodic, constrained energy minimizer for (5.1). We merely summarize the conclusion.

Theorem 5.3 (Orbital stability of RLW constrained energy minimizers). *Let $1/3 < \alpha \leq 2$ and let $u_0(\cdot; c_0, a_0) \in H_{per}^{\alpha/2}([0, T])$ minimize H , defined in (5.2), constrained to conservations of P and M , defined in (5.3) and (5.4), respectively, for some $c_0 \neq 0$, $a_0 \in \mathbb{R}$ and for some $T > 0$.*

Then for all $\varepsilon > 0$ sufficiently small there exists a constant $C(\varepsilon) > 0$ such that: if $\phi \in X$ and $\|\phi\|_X \leq \varepsilon$ and if $u(\cdot, t)$ is a solution of (5.1) for some time interval with the initial condition $u(\cdot, 0) = u_0 + \phi \in X$, satisfying

$$P(u_0 + \phi) = P(u_0) \quad \text{and} \quad M(u_0 + \phi) = M(u_0),$$

then $u(\cdot, t)$ may be continued to a solution for all $t > 0$ such that

$$\sup_{t>0} \inf_{x_0 \in \mathbb{R}} \|u(\cdot, t) - u_0(\cdot - x_0)\|_X \leq C\|\phi\|_X.$$

If, in addition, the matrix $\begin{pmatrix} M_a(u(\cdot; c, a)) & P_a(u(\cdot; c, a)) \\ M_c(u(\cdot; c, a)) & P_c(u(\cdot; c, a)) \end{pmatrix}$ is nonsingular at $u_0(\cdot; c_0, a_0)$ then the same conclusion holds for all $\phi \in X$ such that $\|\phi\|_X \leq \varepsilon$.

6. REMARK ON LINEAR INSTABILITY

We complement the nonlinear stability result in Section 4 (and Theorem 5.3) by discussing the linear instability of periodic traveling waves for KdV type equations

$$(1.1) \quad u_t - \mathcal{M}u_x + f(u)_x = 0.$$

Here \mathcal{M} is a Fourier multiplier defined as $\widehat{\mathcal{M}u}(\xi) = m(\xi)\hat{u}(\xi)$, satisfying

$$(6.1) \quad C_1|\xi|^\alpha \leq m(\xi) \leq C_2|\xi|^\alpha \quad \text{for } |\xi| \gg 1$$

for some $\alpha \geq 1$ and some $C_1, C_2 > 0$, while $f : \mathbb{R} \rightarrow \mathbb{R}$ is C^1 ,

$$(6.2) \quad f(0) = f'(0) = 0 \quad \text{and} \quad \lim_{u \rightarrow \infty} \frac{f(u)}{u} = \infty.$$

Clearly (2.1) fits into the framework.

We assume that (1.1) admits a smooth, four-parameter family of periodic traveling waves, written $u = u(\cdot - x_0; c, a, T)$, where c and a form an open set in \mathbb{R}^2 , $x_0 \in \mathbb{R}$ is arbitrary (and hence we may mod it out), $T_0 < T < \infty$ for some $T_0 > 0$, and where u is T -periodic, satisfying by quadrature that

$$(1.2) \quad \mathcal{M}u - f(u) + cu + a = 0$$

for some c, a (in the sense of distributions). For a broad class of dispersion operators and nonlinearities, the existence of periodic traveling waves of (1.1) may follow from, for example, the mountain pass theorem applied to a suitable variational problem whose Euler-Lagrange equation is (1.2). We assume that (1.1) possesses conserved quantities

$$P(u) = \int_0^T \frac{1}{2} u^2 dx$$

and $M(u) = \int_0^T u dx$, interpret as the momentum and the mass, respectively.

The goal of this section is to derive linear instability criteria for periodic traveling waves of (1.1), which do not necessarily arise as constrained minimizers. In light of Theorem 4.1, a constrained minimizer for (1.1) is expected to be nonlinearly stable under the flow induced by (1.1) under certain assumptions.

Linearizing (1.1) about a nontrivial, periodic traveling wave $u(\cdot; c, a, T)$ in the frame of reference moving at the speed c , we are led to

$$(6.3) \quad v_t = \partial_x L(u)v, \quad \text{where } L(u) = \mathcal{M} - f'(u) + c.$$

We then seek solutions of the form $v(x, t) = e^{\mu t}v(x)$ to arrive at the spectral problem

$$(6.4) \quad \mu v = \partial_x L(u)v,$$

where concerning ourselves with same period perturbations as the underlying wave we consider $L(u)$ acting on $L_{per}^2([0, T])$ with the domain $H_{per}^\alpha([0, T])$. Intuitively u is said *linearly unstable* if the spectrum of $L(u)$ intersects the open right half plane of \mathbb{C} .

Theorem 6.1 (Linear instability). *Let $u(\cdot; c, a, T)$ for some $c, a \in \mathbb{R}$ and for some $T > T_0 > 0$ be a periodic traveling wave of (1.1) under the assumptions (6.1) and (6.2). Assume that the associated linearized operator $L(u)$, acting on $L_{per}^2([0, T])$, is nondegenerate, i.e.*

$$(6.5) \quad \ker(L(u; c, a, T)) = \text{span}\{u_x\}.$$

Then (6.3) admits a nontrivial solution of the form $e^{\mu t}v(x)$, where $v \in H_{per}^\alpha([0, T])$ and $\mu > 0$, if

- (1) $n_-(L(u))$ is odd and $P_c(u(\cdot; c, a, T)) < 0$, or
- (2) $n_-(L(u))$ is even and $P_c(u(\cdot; c, a, T)) > 0$.

Recall that $n_-(L(u))$ denotes the number of negative eigenvalues of $L(u)$ acting on $L_{per}^2([0, T])$.

A thorough proof is found in [Lin08, Section 4] albeit in the solitary wave setting. Here we merely hit the main points.

It is readily seen that (6.4) admits a nontrivial solution in $H_{per}^\alpha([0, T])$ for some $\mu > 0$, namely a purely growing mode, if and only if

$$A^\mu := c - \frac{c\partial_x}{\mu - c\partial_x}(\mathcal{M} - f'(u))$$

enjoys a nontrivial kernel in $H_{per}^\alpha([0, T])$. The spectra of A^μ are shown to lie in the right half plane of \mathbb{C} for $\mu > 0$ sufficiently large, while A^μ converges to $L(u)$ strongly in $L_{per}^2([0, T])$ as $\mu \rightarrow 0+$. Eigenvalues of A^μ near $\mu = 0$ in the left half plane are then studied from those of $L(u)$ via the moving kernel method. Specifically (6.5) ensures that for $\mu > 0$ sufficiently small there exists a unique eigenvalue e_μ of A^μ in the vicinity of the origin that depends upon μ analytically. A lengthy yet explicit calculation reveals that

$$\lim_{\mu \rightarrow 0+} \frac{e_\mu}{\mu} = 0 \quad \text{and} \quad \lim_{\mu \rightarrow 0+} \frac{e_\mu}{\mu^2} = -P_c(u(\cdot; c, a, T)).$$

The linear instability assertion therefore follows since if A^μ has an odd number of eigenvalues in the left half plane of \mathbb{C} , signaling that the spectrum of A^μ crosses the origin at some $\mu > 0$, then a purely growing mode is found.

The proof works *mutatis mutandis* for a broad class of nonlinear dispersive equations including equations of RLW type, but for which the instability “index” is more involved than $P_c(u(\cdot; c, a, T))$; details are discussed in [Lin11].

To conclude, we will contrast Theorem 6.1 as it applies to (2.1) with the stability result in Theorem 4.1 near the large period asymptotics. It is not immediately obvious how Theorem 4.1 and Theorem 6.1 complement each other since Theorem 4.1 is variational in nature, appealing merely to the kernel property of $L(u)$, whereas Theorem 6.1 is in terms of the spectral information of $L(u)$.

We recall from Remark 2.4 that in the range $\alpha > 1/2$ periodic traveling waves of (2.1) constructed as constrained energy minimizers through Proposition 2.2 are

expected to tend to the stable solitary wave as $T \rightarrow \infty$ and $a \rightarrow 0$ simultaneously. Thanks to the scaling invariance (2.6), by the way, (2.7) remains invariant under

$$u(\cdot; c, a, T) \mapsto c^{-1}u(\cdot; 1, c^{-2}a, c^{-1/\alpha}T).$$

Accordingly we may take without loss of generality $c = 1$ and we find that

$$P(1, a, T), M(1, a, T), P_c(1, a, T), M_c(1, a, T) = O(1)$$

for $|a|$ sufficiently small and T sufficiently large; see [BH12] for the detail. Differentiating (2.10) with respect to a and evaluating at a periodic wave near the solitary wave limit, furthermore, we use (2.14) to obtain that

$$M_a = -T + 2M_c = -T + O(1) < 0.$$

Since an explicit calculation reveals that $P_c(u(\cdot; c, a, T)) > 0$ for $|a|$ sufficiently small and T sufficiently large (see also [BH12]), it then follows that

$$M_a P_c - M_c P_a = M_a P_c - M_c^2 = -P_c T + O(1) < 0$$

near the solitary wave limit. In view of (3.3), therefore, we conclude that

$$n_-(L(u(\cdot; c, a, T))) = 1 \quad \text{and} \quad P_c(u(\cdot; c, a, T)) > 0$$

for $|a|$ sufficiently small and for T sufficiently large. Consequently Theorem 6.1 is unable to conclude linear instability of constrained energy minimizers for (2.1) near the solitary wave limit, which is consistent with the stability result in Theorem 4.1. As a matter of fact one may directly appeal to [GSS87], for example, to show that such constrained energy minimizers with large periods and small $a = 0$ are orbitally stable under the flow induced by (2.1).

Acknowledgements. VMH is supported by the National Science Foundation under grant No. DMS-1008885, the University of Illinois at Urbana-Champaign under the Campus Research Board grant No. 11162 and by the Alfred P. Sloan Foundation. MJ gratefully acknowledges support from the National Science Foundation under NSF grant DMS-1211183, and from the University of Kansas General Research Fund under allocation 2302278. The authors thank Zhiwu Lin for sharing his preliminary report [Lin11].

REFERENCES

- [AP07] Jamie Angulo Pava, *Nonlinear stability of periodic traveling wave solutions to the Schrödinger and the modified Korteweg-de Vries equations*, J. Diff. Eq. **235** (2007), 1–30.
- [APBS06] Jaime Angulo Pava, Jerry L. Bona, and Marcia Scialom, *Stability of cnoidal waves*, Adv. Differential Equations **11** (2006), 1321–1374.
- [APN08] Jamie Angulo Pava and Fábio M. A. Natali, *Positivity properties of the Fourier transform and the stability of periodic traveling-wave solutions*, SIAM J. Math. Anal. **40** (2008), 1123–1151.
- [ASB11] Jamie Angulo, Márcia Scialom, and Carlos Banquet, *The regularized Benjamin-Ono and BBM equations: Well-posedness and nonlinear stability*, J. Diff. Eq. **250** (2011), 4011–4036.
- [AT91] Charles J. Amick and John F. Toland, *Uniqueness and related analytic properties for the Benjamin-Ono equation – a nonlinear Neumann problem in the plane*, Acta math. **167** (1991), 107–126.
- [BBM72] T. B. Benjamin, J. L. Bona, and J. J. Mahony, *Model equations for long waves in nonlinear dispersive systems*, Philos. Trans. Roy. Soc. London Ser. A **272** (1972), 47–78.

- [Ben67] T. Brook Benjamin, *Internal waves of permanent form in fluids of great depth*, J. Fluid Mech. **29** (1967), 559–592.
- [Ben72] ———, *The stability of solitary waves*, Proc. Roy. Soc. London Ser. A **328** (1972), 153–183.
- [BH12] Jared C. Bronski and Vera Mikyoung Hur, *Modulational instability and variational structure*, preprint.
- [BJK11] Jared C. Bronski, Mathew A. Johnson, and Todd Kapitula, *An index theorem for the stability of periodic traveling waves of KdV type*, Proc. Royal Soc. A **141** (2011), 1141–1173.
- [BK04] Rodrigo Bañuelos and Tadeusz Kulczycki, *The Cauchy process and the Steklov problem*, J. Funct. Anal. **211** (2004), 355–423.
- [Bon75] Jerry L. Bona, *On the stability theory of solitary waves*, Proc. Roy. Soc. London Ser. A **344** (1975), 363–374.
- [Bou77] Joseph Boussinesq, *Essai sur la théorie des eaux courants*, Mem. prés. div. Sav. Acad. Sci. Inst. Fr. **23** (1877), 1–680.
- [BSS87] J. L. Bona, P. E. Souganidis, and W. A. Strauss, *Stability and instability of solitary waves of Korteweg-de Vries type*, Proc. Roy. Soc. London Ser. A **411** (1987), 395–412.
- [BT09] Jerry L. Bona and Nikolay Tzvetkov, *Sharp well-posedness results for the BBM equation*, DCDS A **23** (2009), 1241–1252.
- [CKS⁺03] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, *Sharp global well-posedness for KdV and modified KdV on \mathbb{R} and \mathbb{T}* , J. Amer. Math. Soc. **16** (2003), 705–749 (electronic).
- [CSS08] L. Caffarelli, S. Salsa, and L. Silvestre, *Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional laplacian*, Invent. Math. **171** (2008), 425–461.
- [DK10] Bernard Deconinck and Todd Kapitula, *The orbital stability of the cnoidal waves of the Korteweg-de Vries equation*, Phys. Letters A. **374** (2010), 4018–4022.
- [DN10] B. Deconinck and M. Nivala, *The stability analysis of the periodic traveling wave solutions of the mKdV equation*, Studies in Applied Mathematics **126** (2010), 17–48.
- [FL12] Rupert L. Frank and Enno Lenzmann, *Uniqueness and nondegeneracy of ground states for $(-\Delta)^2 Q + Q - Q^{\alpha+1} = 0$ in \mathbb{R}* , Acta Math. (2012).
- [GSS87] M. Grillakis, J. Shatah, and W. Strauss, *Stability theory of solitary waves in the presence of symmetry. I, II*, J. Funct. Anal. **74** (1987), 160–197, 308–348.
- [Hur12] Vera Mikyoung Hur, *On the formation of singularities for surface water waves*, CPAA, special issue on hydrodynamic model equations **11** (2012), 1465–1474.
- [Joh09] Mathew A. Johnson, *Nonlinear stability of periodic traveling wave solutions of the generalized Korteweg-de Vries equation*, SIAM J. Math. Anal. **41** (2009), 1921–1947.
- [Joh10] Mathew A. Johnson, *On the stability of periodic solutions of the generalized Benjamin-Bona-Mahony equation*, Phys. D. **19** (2010), 1892 – 1908.
- [Joh12] Mathew A. Johnson, *Stability of small periodic waves in fractional KdV type equations*, preprint.
- [Jos77] R. I. Joseph, *Solitary waves in a finite depth fluid*, J. Phys. A **10** (1977), L224–L227.
- [KdV95] D. J. Korteweg and G. de Vries, *On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves*, Phil. Mag. **39** (1895), 422–443.
- [KMR12] C. E. Kenig, Y. Martel, and L. Robbiano, *Local well-posedness and blow up in the energy space for a class of L^2 critical dispersion generalized Benjamin-Ono equations*, preprint.
- [Kwo89] M. K. Kwong, *Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in \mathbb{R}^n* , Arch. Rational Mech. Anal. **105** (1989), 243–266.
- [Lin08] Zhiwu Lin, *Instability of nonlinear dispersive solitary waves*, J. Funct. Anal. **255** (2008), 1191–1224.
- [Lin11] ———, *Instability of periodic water waves and dispersive waves*, talk at the SIAM Conference on Analysis of PDEs (2011).
- [LY87] E. H. Leib and H. T. Yau, *The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics*, Comm. Math. Phys. **112** (1987), 147–174.
- [MMT97] A. J. Majda, D. W. McLaughlin, and E. G. Tabak, *A one-dimensional model for dispersive wave turbulence*, J. Nonlinear Sci. **7** (1997), 9–44.

- [Mol08] Luc Molinet, *Global well-posedness in L^2 for the periodic Benjamin-Ono equation*, Amer. J. Math. **130** (2008), 635–683.
- [Ono75] Hiroaki Ono, *Algebraic solitary waves in stratified fluids*, J. Phys. Soc. Japan **39** (1975), 1082–1091.
- [RS78] M. Reed and B. Simon, *Methods of modern mathematical physics. IV. Analysis of operators*, Academic Press [Marcourt Brace Jovanovich Publishers], New York, 1978.
- [RS12] Luz Roncal and Pablo Raúl Stinga, *Fractional Laplacian on the torus*, arxiv:1209.6104.
- [SS90] Panayotis E. Souganidis and Walter A. Strauss, *Instability of a class of dispersive solitary waves*, Proc. Roy. Soc. Edinburgh Sect. A **114** (1990), 195–212.
- [Wei87] Michael I. Weinstein, *Existence and dynamic stability of solitary wave solutions of equations arising in long wave propagation*, Comm. Partial Differential Equations **12** (1987), 1133–1173.
- [Whi74] Gerald Beresford Whitham, *Linear and nonlinear waves*, Wiley-Interscience, New York, 1974.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, URBANA, IL 61801 USA

E-mail address: verahur@math.uiuc.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KANSAS, LAWRENCE, KS 66045 USA

E-mail address: matjohn@math.ku.edu